Median voter theorem - continuous choice

In most economic applications voters are asked to make a non-discrete choice - e.g. choosing taxes. In these applications the condition of single-peakedness is related to the curvature properties of the political preference function. If it is quasi-concave in the political choice variable (e.g. the tax rate), then the preference function has a "single peak" and the median voter theorem stated above readily applies.

It is important to note that the condition has to be satisfied by the political preference function.
Median voter theorem - continuous choice

- Is strict concavity (or more mildly quasi-concavity) of political preferences guaranteed by strict concavity (or quasi-concavity) of the underlying preference function (direct utility function)? Not necessarily! The reason is that the latter distinguishes from the direct utility function by the fact that it also reflects optimal individual choices (e.g. labor supply and savings decision) and general equilibrium effects.
Continuous choice - An Example

- $u^i = c + \theta^i b(g)$ with $b' > 0$ and $b'' < 0$.
  $c = l - T$ ($l$ (exogenously given))

- $\theta^i$ measures the preference for public consumption $g$.
  Taxes finance public consumption. With a continuum of individuals whose size is normalized at unity the public budget constraint reads $g = T$. The political preference over the tax rate is $u^i = l - T + \theta^i b(T)$.

- FOC: $-1 + \theta^i b' = 0$.

- SOC: $\theta^i b'' < 0$ (by strict concavity of $b(g)$). $\Rightarrow$ Here, strict concavity of the direct utility function ensures strict concavity (and thus single-peakedness) of political preferences.
Continuous choice - An Example - Modified

- Individuals also derive utility from consuming leisure $l$
- $u^i = c + h(l) + \theta^i b(g)$ with $h' > 0$ and $h'' < 0$.
- Time endowment is normalized to unity. $L + l = 1$.
- Private consumption is $c = w(1 - t)L$.
- Each individual chooses labor supply $L = 1 - l$ such that $u^i = w(1 - t)L + h(1 - L) + \theta^i b(g)$ is maximized taking the tax rate and the level of public consumption as given.
- FOC: $w(1 - t) - h' = 0 \Rightarrow$ Labour supply function $L^*(w(1 - t)). \left(\frac{dL^*}{dt}\right) = \left(\frac{w}{h''}\right) < 0$. 

Continuous choice - An Example - Modified

- The most preferred tax rate follows from maximizing
  \[ u^* = w(1 - t)L^* + h(1 - L^*) + \theta^i b(wtL^*). \]
- FOC: \[ -wL^* + \theta^i b'(wL^* + wt((dL^*)/(dt))) = 0. \]
- SOC: \[ -w((dL^*)/(dt)) + \theta^i b''(wL^* + wt((dL^*)/(dt)))^2 + \theta^i b'(2w((dL^*)/(dt)) + wt((d^2L^*)/(dt^2))) \geq 0 \]
  where \[ ((d^2L^*)/(dt^2)) = ((wh'''((dL^*)/(dt)))/(h'')^2)). \]
- Note: The sign of \( h''' \) is not predetermined
- Although the direct utility function exhibits strong regularity properties, single-peakedness of the induced political preferences is not guaranteed.
Median voter theorem with single-crossing

Single-peakedness is only a sufficient condition for voting cycles not to arise as illustrated in example 2.2.

Example 2.2:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>A</td>
<td>A</td>
<td>C</td>
</tr>
<tr>
<td>.</td>
<td>B</td>
<td>B</td>
<td>A</td>
</tr>
<tr>
<td>3rd</td>
<td>C</td>
<td>C</td>
<td>B</td>
</tr>
</tbody>
</table>

Table 3: Preferences III

A vs. B: 3:0; A vs. C: 2:1 $\Rightarrow$ A wins
A is the Condorcet winner although voter 3’s preferences are not single-peaked (see Figure on next slide).
Median voter theorem with single-crossing

Figure 3: Preferences III
Median voter theorem with single-crossing

A second, more general condition which rules out voting cycles is the condition of single-crossing.

Definition (single-crossing)
Preferences are single-crossing if for any two voters \( i \) and \( j \) \((i < j)\) and for any two alternatives \( q' \) and \( q'' \) with \( q' < q'' \), we have

\[
\begin{align*}
  u_j(q') &> u_j(q'') \Rightarrow u_i(q') > u_i(q'') \\
u_i(q'') &> u_i(q') \Rightarrow u_j(q'') > u_j(q').
\end{align*}
\]

- The key difference to the notion of single-peakedness is that voters (instead of policy alternatives) are ranked according to their "ideology".
Median voter theorem with single-crossing

Example 2.2 (contd.): Assume \( A < B < C \). Are these preferences single-crossing?

- Voters 1 and 3 (1 is the left voter); alternatives A and C (A is the "left" alternative) \( \rightarrow \) no contradiction possible
- Voters 1 and 3 (1 is the left voter); alternatives B and C (B is the "left" alternative) \( \rightarrow \) no contradiction possible
- Voters 1 and 3 (1 is the left voter); alternatives A and B (A is the "left" alternative) \( \Rightarrow u_3(A) > u_3(B) \Rightarrow u_1(A) > u_1(B) \)
  (equivalently for voter 2)
Median voter theorem with single-crossing

Example 2.3: Assume $A < B < C$.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>3rd</td>
<td>C</td>
<td>C</td>
<td>A</td>
</tr>
</tbody>
</table>

Table 4: Preferences IV

Are these preferences single-crossing?
Figure 4: Preferences IV
Median voter theorem with single-crossing

- Voters 1 and 2 (1 is the left voter); alternatives B and C (B is the "left" alternative) → $u_2(B) > u_2(C) \Rightarrow u_1(B) > u_1(C)$
  ✓

- Voters 1 and 2 (1 is the left voter); alternatives A and C (A is the "left" alternative) → $u_2(A) > u_2(C) \Rightarrow u_1(A) > u_1(C)$
  ✓

⇒ No contradiction can be constructed for any other combination of voters and policy alternatives.
⇒ preferences are single-crossing (and single-peaked): B is the Condorcet winner.
The concept of single-crossing allows us to state a 2nd version of the Median voter theorem:

**Theorem (Median voter theorem (single-crossing version))**

*If there is an odd number of voters, individual preferences are single-crossing and the policy space is one-dimensional, then the option most preferred by the voter with a median innate characteristic is the Condorcet winner.*
Median voter theorem with single-crossing

Some final remarks:

- The order $i < j$ is meant to be an invariant order reflecting innate characteristics (such as the political ideology, exogenous productivity or taste for public goods).
- Both concepts (single-peakedness and single-crossing) are logically independent.
- The concept of single-crossing is an ordinal concept. It only requires political preferences to be monotone in the type of the voter.
Some final remarks (cont’d):

- With single-crossing the politically decisive voter is the one who is the median of the invariant order of types.

- Single-peakedness characterizes the politically decisive voter as the one whose preferred alternative is the median of the distribution of the most preferred alternatives.

- The two conditions are only sufficient conditions for a Condorcet winner to exist. It is still possible that a Condorcet winner exists although individual preferences do not satisfy either of both conditions (see example 2.4).
Median voter theorem with single-crossing

Example 2.4: Assume $A < B < C$.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>3rd</td>
<td>C</td>
<td>C</td>
<td>B</td>
</tr>
</tbody>
</table>

Table 5: Preferences V

Voters 2 and 3 (2 is the left voter); alternatives A and B (A is the "left" alternative) ⇒ $u_3(A) > u_3(B) ⇒ u_2(A) > u_2(B) ⇒$ wrong

⇒ Preferences are neither single-crossing nor single-peaked (see Figure on the next slide). But a Condorcet winner exists: A vs. B: 2:1; A vs. C: 2:1
Median voter theorem with single-crossing

Figure 5: Preferences V
Single-crossing - Continuous choice

- If majority voting involves a non-discrete choice, it suffices in many economic applications to check that the marginal rates of substitution are monotone in the voters’ type.

- Whether preferences are single-crossing can be easily verified in the case of an effectively one-dimensional policy space -> see a basic finding in Milgrom (1994) whose relevance for majority voting is illustrated in Gans and Smart (1996): With an effectively one-dimensional policy space political preferences are single-crossing if and only if they satisfy the Spence-Mirrlees condition.
Single-crossing - Continuous choice: Spence-Mirrlees Condition

Assume $u^i(x, y, \theta)$ where $\theta$ is an individual-specific preference parameter.

The Spence-Mirrlees condition requires the marginal rate of substitution between $x$ and $y$ to vary monotonically with the individual’s type $\theta$.

Formally, $\left(\frac{dx}{dy}\right) = -\frac{(\delta u^i)/(\delta y))}{((\delta u^i)/(\delta x))}$ must be either increasing or decreasing in $\theta$ for any combination $(x, y)$. 
Single-crossing - Continuous choice: An Example

- The political preference function is defined over the two policy variables $T$ and $g$, i.e. $u^i = I - T + \theta^i b(g)$.

- Effectively, the policy problem is one-dimensional since $T$ and $g$ are uniquely linked via the public budget constraints, $g = T$. Thus, $g$ is a function of $T$, i.e. $g(T)$.

$$MRS(T, g) = \left(\frac{dT}{dg}\right) = -\left(\frac{\delta u^i}{\delta g}\right) / \left(\frac{\delta u^i}{\delta T}\right) = \theta^i b'(g)$$

- $MRS(T, g)$ is increasing in the preference type $\theta^i$ which guarantees single-crossing.
Single-crossing - Continuous choice: An Example - contd.

- Political preferences are defined as
  \[ u^* = w(1 - t)L^* + h(1 - L^*) + \theta_i b(g). \]

- Slope of the indifference curve in \((g, t)\) space is
  \[ \frac{dT}{dg} = -\frac{\delta u^i}{\delta g} \frac{\delta t}{\delta g} = \frac{\theta_i b'(g)}{wL^*} \]

- Since \( b' \) and \( L^* \) are independent of \( \theta_i \), \( MRS(T, g) \) is strictly increasing in the voter’s type \( \theta_i \).

- Note, as shown above the political preference function may not be single-peaked. However, it is single-crossing.

- As a consequence, the single-crossing version of the Median Voter Theorem can be invoked in characterizing the political equilibrium.
Relevance of the Median Voter Theorem


- In a nutshell, they approach the question whether a legislator’s behavior is tied to the median preference of the district the legislator represents.
Gerber and Lewis (2004): Data and Methodology

- The authors estimate preferences of Los Angeles County voters based on a variety of elections in 1992.

- A second source of data are voter choices revealed in a number of state-wide and local ballot measures on various topics ranging from taxation of candies, property tax exemption of home of person who dies while on active military service, to the introduction of congressional term limits.

- The voting record used in the analysis contains a complete enumeration of all the vote choices made by a given voter, as well as identifying information about the legislative district in which the ballot was cast.
Gerber and Lewis (2004): Data and Methodology

- Voters are grouped according to his/her ideology. Voters who support three times the republican candidate out of the four races for legislative office are classified as republicans (similar for democrats). The remaining set of voters are classified as independent voters.

- For each of these subgroups the authors compute the voter preference using data from the state-wide ballot measures.

- The median preference exhibits some interesting properties. It is more to the left when the share of low-income households and the share of high income households becomes larger. The median voter is also more to the left the higher the educational attainment of the population is - see Table 3 GL.
Gerber and Lewis (2004)

**Analysis of Preference Estimates: OLS Regression Coefficients (N=55)**

Dependent Variable: Estimate of Overall Median or Partisan Median Preference

<table>
<thead>
<tr>
<th>Independent Variable</th>
<th>Median Preference</th>
<th>Democratic Median Preference</th>
<th>Independent Median Preference</th>
<th>Republican Median Preference</th>
</tr>
</thead>
<tbody>
<tr>
<td>%Nonwhite</td>
<td>−1.67 (.26)</td>
<td>−1.32 (.20)</td>
<td>−.17 (.17)</td>
<td>.35 (.17)</td>
</tr>
<tr>
<td>%Income &lt;$20,000</td>
<td>−5.28 (.62)</td>
<td>−4.48 (.48)</td>
<td>−3.73 (.40)</td>
<td>−1.94 (.41)</td>
</tr>
<tr>
<td>%Income &gt;$75,000</td>
<td>−3.09 (.92)</td>
<td>−2.87 (.71)</td>
<td>−1.67 (.60)</td>
<td>−.42 (.61)</td>
</tr>
<tr>
<td>%Education &gt;high school</td>
<td>−2.17 (.47)</td>
<td>−2.57 (.36)</td>
<td>−.78 (.30)</td>
<td>.02 (.31)</td>
</tr>
<tr>
<td>%Clinton</td>
<td>−2.37 (.15)</td>
<td>−1.54 (.36)</td>
<td>−.42 (.30)</td>
<td>.87 (.31)</td>
</tr>
<tr>
<td>Constant</td>
<td>4.37 (.39)</td>
<td>3.59 (.30)</td>
<td>2.43 (.25)</td>
<td>1.80 (.26)</td>
</tr>
</tbody>
</table>

Source.—District characteristics are taken from the 1990 U.S. Census. %Clinton is taken from California Secretary of State (1992b).

Note.—The unit of analysis is the legislative district. Standard errors are in parentheses. Bivariate regressions of each of the four dependent variables on %Clinton alone have R²’s of .89, .59, .51, and .02, respectively.
Gerber and Lewis (2004): Results

- The legislators' behavior is inferred from their roll call votes, i.e. from the record of how the district's legislator voted on a piece of legislation.

- If the median voter theorem is valid, the legislator's behavior must follow from the median voter's preference.

- To formally test for it the authors first regress the median preference on the legislator behavior. The validity of the median voter reasoning can be inferred from two sources:
  1. The median preference and legislator behavior should exhibit a certain degree of co-movement, i.e. the coefficient has to be positive and statistically significant. This is a fundamental condition for the median voter theorem to be empirically relevant.
  2. The share of legislative behavior explained by the median preference (measured by $R^2$) should be sufficiently high.
Gerber and Lewis (2004): Results contd.

- 1st regression: The coefficient of median preference is positive. The $R^2$ is 0.37 leaving a significant amount of variation in legislator behavior unexplained.
- 2nd regression: party ideology is included as an explanatory variable. The sign of the coefficient is positive. The fit of the regression increases to $R^2 = 0.92$ with the consequence of rendering the effect of median preference insignificant.
- 3rd regression: Interaction between median preference and the variance within each district included. The $R^2$ is slightly increased to 0.93. Representing the main result of the paper, the interaction term has a negative impact of legislators behavior while the median preference has once again a positive and significant effect on legislator’s behavior.
Gerber and Lewis (2004): Regression Results

**Determinants of Legislator Behavior: OLS Regression Coefficients (N=55)**

Dependent Variable: Legislator’s First-Dimension NOMINATE Score

<table>
<thead>
<tr>
<th>Independent Variable</th>
<th>Model 1</th>
<th>Model 2</th>
<th>Model 3</th>
<th>Model 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Median preference</td>
<td>.87</td>
<td>.09</td>
<td>.75</td>
<td>.86</td>
</tr>
<tr>
<td></td>
<td>(.15)</td>
<td>(.07)</td>
<td>(.28)</td>
<td>(.31)</td>
</tr>
<tr>
<td>Party ideology</td>
<td>1.12</td>
<td>1.07</td>
<td>1.22</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(.06)</td>
<td>(.06)</td>
<td>(.19)</td>
<td></td>
</tr>
<tr>
<td>Median preference × variance</td>
<td></td>
<td>- .29</td>
<td>- .30</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(.12)</td>
<td>(.12)</td>
<td></td>
</tr>
<tr>
<td>Partisan preference</td>
<td>- .49</td>
<td>- .07</td>
<td>- .14</td>
<td>- .13</td>
</tr>
<tr>
<td></td>
<td>(.09)</td>
<td>(.04)</td>
<td>(.05)</td>
<td>(.15)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>.37</td>
<td>.92</td>
<td>.93</td>
<td>.93</td>
</tr>
</tbody>
</table>

*Note.*—Standard errors are in parentheses. Median preference and variance are as described in table 2. Party ideology is the median NOMINATE score of the members of a legislator’s party delegation in his or her chamber. Partisan preference is Democratic median preference for Democratic legislators and is Republican median preference for Republican legislators (there are no Independent legislators in our sample).

**Figure 7: Regression Results**
Gerber and Lewis (2004): Regression Results contd.

In heterogenous districts legislator’s behavior appears to be less related to the median voter preference. Possible explanations:

- Lobbying (policy for campaign contributions; incentives to lobby in heterogenous districts tend to be larger).
- Party loyalty (party re-election concerns can be more easily traded-off against the median preference if voters do not easily detect such a deviation).
2.3 Preferences — Multi-dimensional voting

Figure 8: Separate voting I
2.3 Preferences — Multi-dimensional voting

The example gives rise to two questions:

1. Which additional conditions are necessary for voting not to be cyclic?
2. If voting cycles do not arise how can one characterize the policy outcome?

⇒ The plan of this part is to introduce three voting concepts which exclude voting cycles. These are respectively: *structure-induced equilibrium*, *intermediate preferences* and *probabilistic voting*.
Structure-induced equilibrium

Idea:

- Partitioning of the voting process into a number of uni-dimensional voting stages.
- If conditions underlying either of the two the Median voter theorems are satisfied, the alternative preferred by the median voter is the Condorcet winner along each policy dimension.
- Let’s take an example...
Structure-induced equilibrium

Figure 9: Separate voting II
Structure-induced equilibrium

- Our example could describe a two-chamber parliamentary system, e.g. Bundestag and a State parliament.
- Due to the separating structure imposed on the voting problem, the equilibrium is referred to as a 
  \textit{structure-induced equilibrium}.
Structure-induced equilibrium

- Our example has featured circular indifference curves implying the existence of "political" best responses which are independent of the political choices taken in the other policy dimension ("political" dominant strategies).

The structure-induced equilibrium features two properties:

1. The median voter in each dimension does not change with the choice taken in the other dimension.

2. The Condorcet winner in each dimension is the level of $x$ (or $y$) which is contained in the bliss-point of the respective median voter.

⇒ Both neat properties will break away if indifference curves are not circular (see the example on the next slide).
Structure-induced equilibrium

Figure 10: Separate voting III
Structure-induced equilibrium

Generally, the concept of a structure-induced equilibrium comes in two forms:

1. **Simultaneous voting**: Each parliament takes the decisions of the other parliament as given.

2. **Sequential voting**: One parliament moves first. Thereby, it anticipates the other parliament’s best-response to its own choice. The other parliament moves afterwards taking the political choice of the “leader” as given.

⇒ In contrast to ”circular“ preferences, the voting outcome differs wrt timing for ”non-circular“ preferences! (See figures on next two slides)
Structure-induced equilibrium

Figure 11: Separate voting IV
Structure-induced equilibrium

Figure 12: Separate voting V
Intermediate preferences

Idea:

- Put more restriction on the preferences instead of on the voting process.
- Conflict over a multi-dimensional policy should be represented as a uni-dimensional conflict.
Intermediate preferences

More formally, consider preferences of an individual are $W(q, \alpha^i)$, where $q$ denotes a policy vector and $\alpha^i$ gives the preference parameter of individual $i$. If individual preferences can be rewritten such that

$$W(q, \alpha^i) = J(q) + K(\alpha^i)H(q),$$

where $K(\alpha^i)$ is monotonic in $\alpha^i$, then the individual is said to have intermediate preferences.
Intermediate preferences

- Denote the median of the distribution of preferences types by $\alpha^m$ and the policy choice most preferred by this voter $q(\alpha^m)$.
- If this policy bundle is pitched against any other policy bundle, $q(\alpha^m)$ receives at least half of the votes.
- The fact that $q(\alpha^m)$ is the Condorcet winner is due to the monotonicity of $K(\alpha^i)$. 
Intermediate preferences

Since $q(\alpha^m)$ is the optimal choice for an $\alpha^m$-type voter

$$J(q(\alpha^m)) + K(\alpha^m)H(q(\alpha^m)) \geq J(q) + K(\alpha^m)H(q)$$

is satisfied for any $q \neq q(\alpha^m)$. 
Intermediate preferences

When \( H(q(\alpha^m)) - H(q) > 0 \), the inequality can equivalently be written as

\[
K(\alpha^m) \geq \frac{J(q) - J(q(\alpha^m))}{H(q(\alpha^m)) - H(q)}.
\]

\( \Rightarrow \)
If \( K(\alpha^i) \) is strictly increasing (decreasing), the inequality holds for all voters with \( \alpha^i > (<) \alpha^m \) which guarantees that \( q(\alpha^m) \) receives at least half of the votes. An analogous reasoning applies when \( H(q(\alpha^m)) - H(q) < 0 \).
Intermediate preferences

**Example:** Consider preferences are 
\[ u^i = U(c) + \alpha^i G(q_1) + (1 - \alpha^i)F(q_2), \] 
where

- \( c \) is private consumption
- \( q_1 \) and \( q_2 \) are two types of public expenditures

Gross income is equal to 1 and \( \tau \) are tax revenues such that \( c = 1 - \tau \). With a population normalized to unity the public budget constraint is \( \tau = q_1 + q_2 \).
Intermediate preferences

Example (cont’d): The preferences satisfy the intermediate preference condition with

- \( J(q) = U(1 - q_1 - q_2) + F(q_2) \),
- \( K(\alpha^i) = \alpha^i \) and
- \( H(q) = G(q_1) - F(q_2) \).
Probabilistic voting

- Another way of solving the non-existence problem is to introduce uncertainty from the candidates’ viewpoints.
- This ensures stable equilibria in both one-dimensional policy space (e.g. if preferences are neither single-peaked nor single-crossing) and multi-dimensional policy space.
- Uncertainty overcomes discontinuities (which cause cycling).
- We relegate a thorough discussion of this model to the end of the next section.
References

3. Electoral competition

-based on the script by Marko Köthenbürger and Tobias Seidel-
We want to study electoral competition along the lines of the policy question how the size of government spending is determined.

...and maintain two key assumptions:

1. Politicians are opportunistic (no partisan preferences)
2. Politicians always implement the announced policy (no commitment problem)

⇒ Politicians only maximize their probability of holding office and not a social welfare function!